A 2-generated 2-related group with no non-trivial finite factors

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Abstract

We construct a 2-generated 2-related group without non-trivial finite factors. That answers a question of J. Button.

Problem 1.12 from [KT] (attributed to Magnus and included in the 1965 edition of [KT] by Greendlinger) asks whether the triviality problem for groups given by balanced presentations with $n \geq 2$ generators is decidable. That problem would have an easy solution if every non-trivial n-generated n-related group would have a non-trivial finite factor. Indeed, if that was the case, then in order to check triviality of that group one could simultaneously list its finite factors and all the relations of the group. The group would be non-trivial if it has a non-trivial finite factor and trivial if one can deduce relations $x = 1$ for all generators x (see more details in [KS, Section 2.6]). Unfortunately, it is well known that there are infinite groups with balanced presentations which do not have non-trivial finite factors. In particular, such is Higman's group $\langle a, b, c, d \mid a^b = a^2, b^c = b^2, c^d = c^2, d^a = d^2\rangle$ [Hi]. But for groups having balanced presentations with fewer than 4 generators the answer seems to be not present in the literature. The question whether a 2-generator 2-relator group without non-trivial finite factors exists was asked by Jack Button (we found out about this question from Ian Leary). In this note, we give a positive answer to this question. Our example also has a 3-generated balanced presentation, of course (add a new generator x and a relation $x = 1$).

Theorem 1. *There exists an infinite 2-generated 2-related group with no non-trivial finite factors.*

Proof. Let G be the Baumslag group $\langle a, t \mid a^{a^t} = a^2\rangle$ where x^y denotes $y^{-1}xy$. It is known [Ba] that all finite factors of G are cyclic. Moreover, the image of a is 1 in every finite factor of G.

It will be convenient for us to represent G as an HNN extension $\langle a, b, t \mid a^t = b, a^b = a^2\rangle$ of the Baumslag-Solitar group $H = \langle a, b \mid a^b = a^2\rangle$ with associated subgroups $A = \langle a\rangle$ and $B = \langle b\rangle$ such that $t^{-1}At = B$ in G.

Recall [LS] if an element $g \in G$ is equal to the product $h_0t^{\epsilon_1}h_1 \ldots t^{\epsilon_n}h_n$ where $h_0, \ldots, h_n \in H$ and $\epsilon_i = \pm 1$ ($i = 1, \ldots, n$), then the t-length of g is n. The product is called *reduced* if it has no occurrences of $t^{-1}xt$ with $x \in A$ or tyt^{-1} with $y \in B$. It is called *cyclically reduced* if every cyclic permutation of that product is reduced. By Britton's lemma [LS] two reduced products representing the same element in G have equal t-lengths. The following property holds for the Baumslag group G.

($\star$) *For every integer $n \geq 2$ and $x, y \in H$, we have $t^nyt^{-n} = x$ in G if and only if $x = y = 1$.*

*The authors were supported in part by the NSF grants DMS 0245600 and DMS 0455881. In addition, the research of the first author was supported in part by the Russian Fund for Basic Research grant 05-01-00895, and the research of the second author was supported by a BSF (USA-Israeli) grant.

Indeed, the t-length of the right-hand side is 0, and so $y \in B$, and $tyt^{-1} = z$ in G, where $z \in A$. But also $z \in B$ since $tzt^{-1} = x$ in G . Hence $z \in A \cap B = \{1\}$, and $x = y = 1$.

Let r be any word of the form

$$bt^{u_1}at^{-u_2}bt^{u_3}...at^{-u_l}$$

where the numbers l and u_i satisfy the following conditions:

(1) l is even, and u_i are different integers, $u_i \geq 2$

(2) the total exponent of t, i.e. $u_1 - u_2 + u_3... - u_l$ is 1

(3) $\max(u_1 + u_2 + u_3, u_2 + u_3 + u_4, \dots, u_{l-1} + u_l + u_1, u_l + u_1 + u_2) < \frac{1}{6}\sum_{i=1}^{l} u_i$

(for example one can take $l = 20$, $u_{2i-1} = 100 + 2i - 1$ for $i = 1, \dots, 9$, $u_{19} = 130$, $u_{2i} = 100 + 2i$ for $i = 1, \dots, 10$). Consider the factor-group $K = \langle G \mid r = 1 \rangle$. The images of a and $b = a^t$ vanish in every finite factor-group F of K since the same property holds for G. It follows from (2) and equality $r = 1$ that the image of t in F is also trivial. Therefore K does not have non-trivial finite factors, and it remains to prove that K itself is not trivial.

The product r is cyclically reduced. Let R be the set of all cyclically reduced forms of the conjugate elements of r and r^{-1} in G.

Lemma 2. *(D.Collins [LS] , IV.2.5)*

Let $w = h_1 t^{\epsilon_1} \dots h_n t^{\epsilon_n}$ $(n \geq 1)$ and w' be conjugate cyclically reduced elements in an HNN extension $H^\star$ of a group H ($h_1, \dots, h_n \in H$). Then w' is equal in $H^\star$ to $h^{-1}w^\star h$ for a cyclic permutation $w^\star$ of w and some $h \in H$.

Lemma 3. *Let r_1 and r_2 belong to R and equal in G to some reduced products starting with $t^{\pm u_s}xt^{\mp u_{s+1}}x'$ and $t^{\pm u_s}yt^{\mp u_{s+1}}y'$, respectively, where x, x', y, and y' are nontrivial elements of H, and the subscripts are taken modulo l. Then $r_1 = r_2$ in G.*

Proof. By Lemma 2 and condition (1), r_1 and r_2 are conjugates in G of the same cyclic permutation of r by some elements of H. Hence $r_2 = hr_1h^{-1}$ in G for some $h \in H$. But the factors $t^{\pm u_s}$ must cancel out in the product of $r_1^{-1}r_2$ by the lemma condition. Therefore they also must cancel in the product of reduced r_1 and hr_1h^{-1}. Hence $\dots t^{\mp u_s}ht^{\pm u_s} \in H$. It follows from Condition (1) and Property ($\star$) that $h = 1$, and therefore $r_1 = r_2$. ☐

Lemma 4. *Let $r_1 = vw_1$ and $r_2 = vw_2$ in G, where $r_1, r_2 \in R$, and these products are reduced in G. Assume that a reduced form of v starts with $t^u h_1 t^{\pm u_s} h_2 t^{\mp u_{s+1}} h_3$ for some non-trivial $h_1, h_2, h_3 \in H$ and an integer u (subscripts are taken modulo l). Then $r_1 = r_2$ in G.*

Proof. Denote the prefix $t^u h_1$ of v by z. There are reduced forms of $z^{-1}r_1z$ and of $z^{-1}r_2z$ both started with $t^{\pm u_s}h_2t^{\mp u_{s+1}}h_3$. By Lemma 3, we have $z^{-1}r_1z = z^{-1}r_2z$, and so $r_1 = r_2$ in G. ☐

Proof of the theorem. Let $r_1 \in R$, and assume that r_1 is conjugate of r in G. (The proof is similar, if it is conjugate of r^{-1}.) If r_1 and a word $r_2 \in R$ have a left piece v as in the formulation of Lemma 4, then $r_1 = r_2$ by this lemma. Otherwise v contains at most 3 t-blocks, and so its t-length is less than $\frac{1}{6}|r_1|$ by Condition (3). Hence our presentation $K = \langle G \mid r = 1 \rangle$ satisfies the small cancellation condition $C'(\frac{1}{6})$ by definition (see [LS], V.11). Then by Theorem 11.6 [LS], the canonical homomorphism of H into K is injective. Therefore the group K is infinite, as required.

Remark 5. Note that not only finite, but all torsion homomorphic images of the group K are trivial because the image of a is trivial in every torsion factor-group of G [Ba]. Since every hyperbolic group is residually torsion ([Gr],[Ol]), it follows that the image of K under any homomorphism of K into a hyperbolic group is trivial. On the other hand, K has continuously many normal subgroups, and moreover, it is SQ-universal. To prove this, one can just combine the above small cancellation argument with the constructions from the proofs of theorems 11.3 and 11.7 in [LS].

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